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# POSSIBLE EFFECT OF VARIATION IN THE INTERNAL ENERGY OF THE FREE SURFACE OF A THIN LIQUID LAYER ON ITS WAVY FLOW

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*A formally possible mechanism is considered for the development of longwave perturbations of flow of a thin layer of a heat-conducting viscous liquid with a free boundary, whose characteristic feature is that the Marangoni stresses occurring at the boundary are induced by variations in the internal energy of the interface. The effect of surface internal energy fluctuations on the layer flow in the approximation considered is of a dispersive character, and, in particular, it can facilitate regularization of wave regimes.*

**Introduction.** Recently, some interest has been expressed in problems related to the effect of the interfacial internal energy in liquid–liquid and liquid–gas systems on the temperature and velocity fields in the vicinity of the interface (see, for example, [1]). In most studies of thermocapillary phenomena, the effects related to the variation in the surface internal energy are considered insignificant. However, as calculations show [1], for most liquids, including water, at rather high temperatures and for low-viscosity liquids, these effects may have a substantial impact on dynamics in the vicinity of the interface.

A liquid film layer is a physical system that seems to be quite convenient for both experimental and theoretical studies of this type of phenomenon.

In the present paper, we consider a possible physical mechanism for the influence of variations in the surface internal energy on the temperature field in the vicinity of the free boundary of a liquid film and, ultimately, on the formation of Marangoni stresses and a velocity field. The effects due to thermal expansion of the liquid are not taken into account since the layer thickness is assumed to be rather small.

The flow of a film liquid layer with a free boundary has long been thoroughly studied [2–5], in particular, in relation to various practical applications. Film flows are widely used in highly efficient mass-transfer devices and are the basis of some thermal phenomena and chemical–technological processes (absorption, desorption, cooling, condensation, etc.) [6]. The character of wavy regimes in a film layer can substantially affect the rate of transfer processes through the interface [4, 7], which makes the search for physical factors and effective methods of controlling flow dynamics important. At the same time, film flows are among the simplest, most accessible, and expressive physical systems that illustrate various nonlinear phenomena.

Our analysis is based on a widely known simplified mathematical model that describes weakly nonlinear wavy regimes of film flow — the Kuramoto–Sivashinskii equation (KS-model) [8–10]:

$$\frac{\partial}{\partial t} H + H \frac{\partial}{\partial x} H + \frac{\partial^2}{\partial x^2} H + \frac{\partial^4}{\partial x^4} H = 0.$$

Here $t$, $x$, and $H$ are the scaled time, longitudinal coordinate, and perturbation of the interface, respectively. This model applies for high surface tension of the film free boundary, rather large longitudinal spatial scale of perturbations in the thin layer, and Reynolds numbers of order $O(1)$. Although the solution of this equation does not give a satisfactory quantitative description of the wavy flows observed in experiments, there is qualitative agreement for waves of small amplitudes.

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In the present paper, the physical mechanism used as the basis of the KS-flow model for a thin layer of a viscous liquid is considered with allowance for some thermocapillary effects. The method of multiscale tensions within the limit of asymptotically large effective surface tension and Marangoni number reduces the initial mathematical model to the following evolution equation for the weakly linear dynamics of a free boundary, which is a combination of the Korteweg–de Vries and Kuramoto–Sivashinskii equations:

$$\frac{\partial}{\partial t} H + H \frac{\partial}{\partial x} H + \frac{\partial^2}{\partial x^2} H + D \frac{\partial^3}{\partial x^3} H + \frac{\partial^4}{\partial x^4} H = 0.$$

The term $DH_{xxx}$ in this equation is due to the effect of the internal energy of the film free surface on flow dynamics.

Among the most important features of the periodic solutions of the Kuramoto–Sivashinskii equation is the property that the development of any initial longwave perturbation is of an irregular character: the corresponding numerical solutions are chaotic oscillations in time in the presence of coherent spatial structures [9, 11–13]. According to the assumption of [9], the chaotic nature of the longwave periodic solutions of the Kuramoto–Sivashinskii equations can be responsible for the irregular (IR) behavior of film layers observed in experiments. As shown by numerical studies of the Kuramoto–Sivashinskii and Korteweg–de Vries equations [14–16], the term with the third derivative exerts a regularizing influence on dynamics. The initial data, which develop chaotically for $D = 0$, evolve, for large values of the coefficient $D$, to the limiting regime with a regular sequence of stationary waves of the same shape. This suggests that the thermocapillary mechanism considered in the present paper can exert a regularizing influence on the behavior of the film layer. In addition, some other possibilities of affecting the dynamics of film flow arise.

**1. Formulation of the Problem.** Let us give the initial mathematical model in invariant form. We assume that the behavior of a viscous heat-conducting liquid in a region $\Omega$ bounded by a surface $S$ (solid wall) and a free surface $\Gamma$ adjacent to a gaseous phase is described by the following system of equations and conditions [17].

In the region $\Omega$, the Navier–Stokes and incompressibility equations and the Fourier law are satisfied:

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \mathbf{f}^*, \quad \operatorname{div}\mathbf{v} = 0, \quad \frac{d\theta}{dt} = \chi\Delta\theta.$$

Here $\mathbf{v}(\mathbf{x},t)$ is the velocity vector, $p(\mathbf{x},t)$ is the pressure, $\theta(\mathbf{x},t)$ is the temperature, $\mathbf{x} \in \mathbf{R}^3$ is the radius-vector, $t$ is time, $\mathbf{f}^*$ is the acceleration of gravity, $\rho$, $\nu$, and $\chi$ are the constant density, kinematic viscosity, and thermal diffusivity of the liquid, respectively, and $\nabla$, div, and $\Delta$ are the gradient, divergence, and Laplacian operators, respectively.

On the free boundary ($\mathbf{x} \in \Gamma$) there are [17]

— the kinematic condition of nonpenetration $\mathbf{v} \cdot \mathbf{n} = V_n$,

— the stress-balance condition $P \cdot \mathbf{n} + p^g\mathbf{n} = 2\sigma H\mathbf{n} + \nabla_\Gamma \sigma$,

— the energy-transfer condition $(k\nabla\theta - k^g\nabla\theta^g) \cdot \mathbf{n} = \theta \dfrac{d\sigma}{d\theta} \operatorname{div}_\Gamma \mathbf{v} + \theta \dfrac{d^2\sigma}{d\theta^2} \dfrac{d\theta}{dt}$.

Here $\mathbf{n}$ is a unity vector normal to the surface $\Gamma$, which is outer with respect to the region $\Omega$, $V_n$ is the displacement velocity of the surface $\Gamma$ in the direction of the normal $\mathbf{n}$, $\nabla_\Gamma \equiv \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla)$, $\operatorname{div}_\Gamma \mathbf{v} \equiv \nabla \cdot \mathbf{v} - \mathbf{n}((\mathbf{n} \cdot \nabla)\mathbf{v})$, $P$ is the stress tensor, $H$ is the mean curvature of the surface $\Gamma$, $k$ is the constant thermal conductivity of the liquid, $\sigma > 0$ is the surface-tension coefficient on the free boundary $\Gamma$, and $p^g$, $\theta^g$, and $k^g$ are the pressure, temperature, and constant thermal conductivity of the gaseous phase, respectively. The functions $p^g$ and $\theta^g$ are assumed to be known.

The energy-transfer condition means that the heat flux jump in the normal direction to the surface $\Gamma$ is compensated by the variation in the internal energy of this surface due to variations in the interface area and temperature [17].

Finally, we assume that the standard conditions of attachment and constant temperature are satisfied on the solid wall $S$.

We consider a thin film layer of a viscous heat-conducting incompressible liquid flowing under gravity along the outer surface of a vertical cylindrical pipe of radius $a^*$. Constant temperature $\theta_S^*$ is maintained

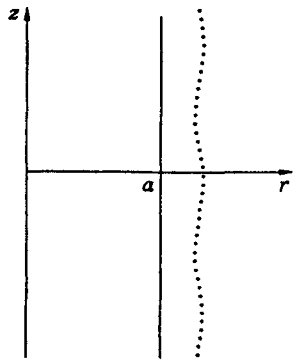


Fig. 1

along the pipe. It is assumed that the temperature dependence of the surface-tension coefficient is linear [this is a good approximation of real dependences for many liquids (water, solutions of organic substances, etc)] in certain ranges of temperature variation [18]): $\sigma = \sigma^{(0)} - æ(\theta - \theta^{(0)})$, where $\sigma^{(0)} = \text{const}$ is the surface tension at temperature $\theta^{(0)}$ and $æ = \text{const}$ is a temperature coefficient; below, we assume that $\theta^{(0)} = \theta_S^*$.

Let $(r,\ \varphi,\ z)$ be cylindrical coordinates (radial, angular, and axial coordinates and the $z$ axis is opposite to the gravity force) and $(u,\ v,\ w)$ be the velocity field.

As the basic process, we examine the laminar isothermal flow of the layer in the region $\Omega_0 = \{a^* \leqslant r \leqslant a^* + h_0^*,\ -\pi \leqslant \varphi \leqslant \pi,\ -\infty < z < \infty\}$ ($h_0^* = \text{const}$). The scale coefficients for the spatial variables, velocity, time, pressure, and temperature are $h_0^*$, $w^* = f^* h_0^{*2}/2\nu$, $h_0^*/w^*$, $\rho w^{*2}$, and $\theta_S^*$, respectively ($f^*$ is the acceleration of gravity). We assume that $a^*/h_0^* = a$ and study the dimensionless problem. The general view of the system considered is shown schematically in Fig.1.

We write the initial mathematical model in the cylindrical coordinates. In the region $\{a < r < a + h(t,\varphi,z),\ -\pi \leqslant \varphi \leqslant \pi,\ -\infty < z < \infty\}$, the following conditions must be satisfied:

$$u_t + u u_r + \frac{v}{r} u_\varphi + w u_z - \frac{v^2}{r} = -p_r + \frac{1}{\mathrm{Re}}\Big(u_{rr} + \frac{1}{r^2} u_{\varphi\varphi} + u_{zz} + \frac{1}{r} u_r - \frac{2}{r^2} v_\varphi - \frac{u}{r^2}\Big); \tag{1.1}$$

$$v_t + u v_r + \frac{v}{r} v_\varphi + w v_z + \frac{uv}{r} = -\frac{1}{r} p_\varphi + \frac{1}{\mathrm{Re}}\Big(v_{rr} + \frac{1}{r^2} v_{\varphi\varphi} + v_{zz} + \frac{1}{r} v_r + \frac{2}{r^2} u_\varphi - \frac{v}{r^2}\Big); \tag{1.2}$$

$$w_t + u w_r + \frac{v}{r} w_\varphi + w w_z = -p_z + \frac{1}{\mathrm{Re}}\Big(w_{rr} + \frac{1}{r^2} w_{\varphi\varphi} + w_{zz} + \frac{1}{r} w_r\Big) - \frac{2}{\mathrm{Re}}; \tag{1.3}$$

$$u_r + \frac{u}{r} + \frac{1}{r} v_\varphi + w_z = 0; \tag{1.4}$$

$$\theta_t + u\theta_r + \frac{v}{r}\theta_\varphi + w\theta_z = \frac{1}{\mathrm{Pe}}\Big(\theta_{rr} + \frac{1}{r^2}\theta_{\varphi\varphi} + \theta_{zz} + \frac{1}{r}\theta_r\Big). \tag{1.5}$$

The boundary conditions have the form

$$u = 0, \quad v = 0, \quad w = 0, \quad \theta = 1 \quad \text{for} \quad r = a. \tag{1.6}$$

For $r = a + h(t,\varphi,z)$, the following relations hold:

- The balance condition for normal stresses

$$\begin{gathered} -\mathrm{Re}\,(p - p^g) + 2\Big(1 + \frac{1}{r^2} h_\varphi^2 + h_z^2\Big)^{-1}\Big(u_r - \frac{1}{r} h_\varphi\Big(\frac{1}{r} u_\varphi + v_r - \frac{1}{r} v\Big) - h_z(w_r + u_z) \\ + \frac{1}{r} h_\varphi h_z\Big(v_z + \frac{1}{r} w_\varphi\Big) + \frac{1}{r^2} h_\varphi^2\Big(\frac{1}{r} v_\varphi + \frac{1}{r} u\Big) + h_z^2 w_z\Big) = (\mathrm{We} + \mathrm{Mn}\,(\theta - 1))\Big(-\frac{2}{r^3} h_\varphi^2 \\ -\frac{1}{r}(1 + h_z^2) + \frac{1}{r^2} h_{\varphi\varphi}(1 + h_z^2) - \frac{2}{r^2} h_\varphi h_z h_{\varphi z} + h_{zz}\Big(1 + \frac{1}{r^2} h_\varphi^2\Big)\Big)\Big(1 + \frac{1}{r^2} h_\varphi^2 + h_z^2\Big)^{-3/2}; \end{gathered} \tag{1.7}$$

• The balance condition for tangential stresses

$$\Big(1+\frac{1}{r^2}h_\varphi^2+h_z^2\Big)^{-1/2}\Big(\frac{2}{r}h_\varphi u_r+\Big(1-\frac{1}{r^2}h_\varphi^2\Big)\Big(\frac{1}{r}u_\varphi+v_r-\frac{1}{r}v\Big)-h_z\Big(v_z+\frac{1}{r}w_\varphi\Big)$$
$$-\frac{1}{r}h_zh_\varphi(w_r+u_z)-\frac{2}{r}h_\varphi\Big(\frac{1}{r}v_\varphi+\frac{1}{r}u\Big)\Big)=\mathrm{Mn}\Big(\frac{1}{r}\theta_rh_\varphi+\frac{1}{r}\theta_\varphi\Big); \qquad (1.8)$$

$$\Big(1+\frac{1}{r^2}h_\varphi^2+h_z^2\Big)^{-1/2}\Big(2h_zu_r+(1-h_z^2)(w_r+u_z)-\frac{1}{r}h_\varphi h_z\Big(\frac{1}{r}u_\varphi+v_r-\frac{1}{r}v\Big)$$
$$+\frac{1}{r}h_\varphi\Big(v_z+\frac{1}{r}w_\varphi\Big)-2h_zw_z\Big)=\mathrm{Mn}\,(\theta_rh_z+\theta_z); \qquad (1.9)$$

• The energy-transfer condition

$$\Big(1+\frac{1}{r^2}h_\varphi^2+h_z^2\Big)^{-1/2}\Big(\theta_r-\frac{1}{r^2}h_\varphi\theta_\varphi-h_z\theta_z-\frac{k^g}{k}\Big(\theta_r^g-\frac{1}{r^2}h_\varphi\theta_\varphi^g-h_z\theta_z^g\Big)\Big)$$
$$=\mathrm{E}\,\theta\Big(1+\frac{1}{r^2}h_\varphi^2+h_z^2\Big)^{-1}\Big(u_r-\frac{1}{r^2}h_\varphi u_\varphi-h_zu_z+\frac{1}{r^2}h_\varphi v-\frac{1}{r}h_\varphi v_r$$
$$+\frac{1}{r^3}h_\varphi^2v_\varphi+\frac{1}{r}h_\varphi h_zv_z+\frac{1}{r^3}h_\varphi^2u-h_zw_r+\frac{1}{r^2}h_\varphi h_zw_\varphi+h_z^2w_z\Big); \qquad (1.10)$$

• The kinematic nonpenetration condition

$$u=h_t+\frac{v}{r}h_\varphi+wh_z. \qquad (1.11)$$

Here $\mathrm{Re}=w^*h_0^*/\nu$, $\mathrm{Pe}=w^*h_0^*/\chi$, $\mathrm{We}=\sigma^{(0)}/(\rho\nu w^*)$, $\mathrm{Mn}=-æ\theta_S^*/(\rho\nu w^*)$, $\mathrm{E}=æw^*/k$, and $p^g$ and $\theta^g$ are specified functions.

The basic laminar flow is defined as follows:

$$u_0=0,\ \ v_0=0,\ \ w_0=(r^2-a^2)/2-(1+a)^2\ln(r/a),\ \ p_0=\mathrm{const},\ \ \theta_0=1,\ \ h_0=1. \qquad (1.12)$$

Our task is to study the development of perturbations for the basic state (1.12) at the weakly nonlinear stage of evolution of the fluctuations.

**2. Derivation of the Amplitude Equation.** In this section, reducing the initial problem by the method of multiscale tensions [8, 10] we derive the evolution equation describing the longwave regimes on the free surface of the layer.

**Choice of Spatial and Time Scales.** In [10], the choice of scales for the dependent and independent variables was based on the following considerations. The linear dispersion relation obtained from analysis of the linear stability of flow of a viscous liquid layer down a vertical plane wall ignoring thermocapillary effects leads to the following estimates [10] for unsteady perturbation of the form $\exp(\lambda t+in\hat{z}+il\hat{y})$ ($\hat{y}$ and $\hat{z}$ are the horizontal and vertical coordinates):

$$n\sim l\sim \mathrm{We}^{-1/2},\quad \mathrm{Real}\,(\lambda)\sim \mathrm{We}^{-1},\quad \mathrm{Im}\,(\lambda)\sim \mathrm{We}^{-1/2}\quad \text{for}\quad \mathrm{We}\gg 1,\quad \mathrm{Re}=O(1). \qquad (2.1)$$

We assume that the longitudinal spatial scale of the flow in cylindrical geometry agrees with estimate (2.1). From (2.1) it also follows that the characteristic spatial scale of transverse perturbation $2\pi/l\sim \mathrm{We}^{1/2}$. The wall curvature is assumed to exert the strongest influence on the flow properties when the number of transverse waves $O(al)$ is finite. Hence, we obtain $a\sim \mathrm{We}^{1/2}$.

Considering problem (1.1)–(1.11) linearized on solution (1.12) and expanding the perturbations and the quantity $\lambda$ in a series in a small parameter — the wave number $n$ — we find that estimates (2.1) remain the same if $\mathrm{Mn}=O(\mathrm{We}^{1/2})$, $\mathrm{Pe}=O(1)$, and $\mathrm{E}=O(1)$.

Next we consider secondary flows whose characteristic spatial scales in the longitudinal and transverse directions are of order $O(\varepsilon^{-1})$, $0<\varepsilon\ll 1$ ($\varepsilon$ is the smallness parameter); in this case, $\mathrm{We}=O(\varepsilon^{-2})$.

We set [10]

$$X = r - a, \quad Z = \varepsilon z, \quad \alpha = \varepsilon a, \quad Y = \alpha\varphi \quad (-\alpha\pi \leqslant Y \leqslant \alpha\pi), \quad \frac{\partial}{\partial t} \to \frac{\varepsilon\partial}{\partial\tau} + \frac{\varepsilon^2\partial}{\partial T}; \tag{2.2}$$

$$u = \sum_{n=1} \varepsilon^n U_n(X,Y,Z,\tau,T), \quad w = w_0(X,\varepsilon) + \sum_{n=1} \varepsilon^n W_n(X,Y,Z,\tau,T),$$

$$p = p_0(\varepsilon) + \sum_{n=1} \varepsilon^n P_n(X,Y,Z,\tau,T), \tag{2.3}$$

$$\theta = 1 + \sum_{n=1} \varepsilon^n \Theta_n(X,Y,Z,\tau,T), \quad h = 1 + \sum_{n=1} \varepsilon^n H_n(Z,Y,\tau,T).$$

Note that $w_0(X,\varepsilon) = X^2 - 2X - \varepsilon(X^3/3 - X^2 + X)/\alpha + O(\varepsilon^2)$.

We assume that the orders of magnitude of the dimensionless parameters of the problem are defined by

$$\mathrm{We} = O(\varepsilon^{-2}), \quad \mathrm{Mn} = O(\varepsilon^{-1}), \quad \mathrm{Re} = O(1), \quad \mathrm{Pe} = O(1), \quad \mathrm{E} = O(1). \tag{2.4}$$

We denote $\overline{\mathrm{Mn}} = \mathrm{Mn}\,\varepsilon$ and $\overline{\mathrm{We}} = \mathrm{We}\,\varepsilon^2$.

**Influence of Temperature Perturbations in the Gas Phase.** In most cases, the thermal diffusivity of gases is an order of magnitude lower than that for liquids [18], and, hence, it is quite natural to assume that $k^g/k \leqslant O(\varepsilon)$. We set

$$\theta^g = 1 + \sum_{n=1} \varepsilon^n \Theta_n^g(X,Y,Z,\tau,T), \qquad X \geqslant 1.$$

The characteristic spatial scale in the radial direction in the region occupied by the gas phase coincides with the scales in the longitudinal and transverse directions. Therefore, in the region adjacent to the film layer ($X \geqslant 1$), we have $\partial/\partial X = O(\varepsilon)$. With allowance for the above-mentioned assumptions and remarks, in the case considered, the temperature perturbation in the gas phase has little effect on flow dynamics in the film layer.

At the same time, one can believe that temperature fluctuations in the region adjacent to the thin layer are suppressed by the isothermal gas flow.

**Sequence of Approximation Problems.** Substituting (2.2) and (2.3) into (1.1)–(1.11) and taking into account (2.4), we obtain the following sequence of approximation problems.

In the first order,

$$\frac{\partial P_1}{\partial X} = \frac{1}{\mathrm{Re}}\frac{\partial^2 U_1}{\partial X^2}, \quad \frac{\partial^2 V_1}{\partial X^2} = 0, \quad \frac{1}{\mathrm{Re}}\frac{\partial^2 W_1}{\partial X^2} = 2(X-1)U_1,$$
$$\frac{\partial U_1}{\partial X} = 0, \quad \frac{\partial^2 \Theta_1}{\partial X^2} = 0 \quad \text{for} \quad 0 < X < 1; \tag{2.5}$$

$$U_1 = V_1 = W_1 = \Theta_1 = 0 \quad \text{for} \quad X = 0; \tag{2.6}$$

$$P_1 = -\frac{\overline{\mathrm{We}}}{\mathrm{Re}}\Big(\frac{1}{\alpha^2}H_1 + \nabla^2 H_1\Big) + \frac{2}{\mathrm{Re}}\frac{\partial U_1}{\partial X}, \quad \frac{\partial V_1}{\partial X} = \overline{\mathrm{Mn}}\frac{\partial \Theta_1}{\partial Y},$$
$$2H_1 + \frac{\partial W_1}{\partial X} = \overline{\mathrm{Mn}}\frac{\partial \Theta_1}{\partial Z}, \quad \frac{\partial \Theta_1}{\partial X} = \mathrm{E}\frac{\partial U_1}{\partial X}, \quad U_1 = 0 \quad \text{for} \quad X = 1. \tag{2.7}$$

Here $\nabla^2 \equiv \partial^2/\partial Y^2 + \partial^2/\partial Z^2$. Problem (2.5)–(2.7) has the solution

$$U_1 = 0, \quad V_1 = 0, \quad W_1 = -2H_1X, \quad \Theta_1 = 0, \quad P_1 = -\frac{\overline{\mathrm{We}}}{\mathrm{Re}}\Big(\frac{1}{\alpha^2}H_1 + \nabla^2 H_1\Big). \tag{2.8}$$

In the second order, Eqs. (1.2)–(1.6) and (1.8)–(1.11) have the form

$$-\frac{\partial P_1}{\partial Y} + \frac{1}{\mathrm{Re}}\frac{\partial^2 V_2}{\partial X^2} = 0,$$

$$\frac{\partial W_1}{\partial \tau} + 2(X-1)U_2 + X(X-2)\frac{\partial W_1}{\partial Z} = -\frac{\partial P_1}{\partial Z} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 W_2}{\partial X^2} + \frac{1}{\alpha}\frac{\partial W_1}{\partial X}\right), \tag{2.9}$$

$$\frac{\partial U_2}{\partial X} + \frac{\partial W_1}{\partial Z} = 0, \quad \frac{\partial^2 \Theta_2}{\partial X^2} = 0 \quad \text{for} \quad 0 < X < 1;$$

$$U_2 = V_2 = W_2 = \Theta_2 = 0 \quad \text{for} \quad X = 0; \tag{2.10}$$

$$\frac{\partial V_2}{\partial X} = \overline{\mathrm{Mn}}\,\frac{\partial \Theta_2}{\partial Y}, \quad \frac{\partial W_2}{\partial X} + 2H_2 = \overline{\mathrm{Mn}}\,\frac{\partial \Theta_2}{\partial Z}, \quad \frac{\partial \Theta_2}{\partial X} = \mathrm{E}\frac{\partial U_2}{\partial X},$$
$$\frac{\partial H_1}{\partial \tau} - \frac{\partial H_1}{\partial Z} = U_2 \quad \text{for} \quad X = 1. \tag{2.11}$$

Problem (2.9)-(2.11) with allowance for (2.8) has the solution

$$H_1 = H_1(\zeta, T), \quad U_2 = X^2\frac{\partial H_1}{\partial \zeta}, \quad \Theta_2 = 2\mathrm{E}\,X\frac{\partial H_1}{\partial \zeta},$$

$$V_2 = -\frac{1}{2}\overline{\mathrm{We}}\,X(X-2)\left(\frac{1}{\alpha^2}\frac{\partial H_1}{\partial Y} + \nabla^2\frac{\partial H_1}{\partial Y}\right) + 2\overline{\mathrm{Mn}}\,\mathrm{E}\,X\frac{\partial^2 H_1}{\partial \zeta \partial Y}, \tag{2.12}$$

$$W_2 = \mathrm{Re}\left(\frac{1}{6}X^4 - \frac{2}{3}X^3 + \frac{4}{3}X\right)\frac{\partial H_1}{\partial \zeta} - \frac{1}{2}\overline{\mathrm{We}}\,X(X-2)\left(\frac{1}{\alpha^2}\frac{\partial H_1}{\partial \zeta} + \nabla^2\frac{\partial H_1}{\partial \zeta}\right)$$
$$+\frac{1}{\alpha}X(X-2)H_1 - 2XH_2 + 2\overline{\mathrm{Mn}}\,\mathrm{E}\,X\frac{\partial^2 H_1}{\partial \zeta^2},$$

where $\zeta = Z + 2\tau$.

In the third order, the kinematic condition (1.11) has the form

$$\frac{\partial H_1}{\partial T} + \frac{\partial H_2}{\partial \zeta} + W_1\frac{\partial H_1}{\partial \zeta} - \frac{1}{3\alpha}\frac{\partial H_1}{\partial \zeta} = U_3 + \frac{\partial U_2}{\partial X}H_1 \qquad (X = 1). \tag{2.13}$$

This condition is the basis of the required evolution equation. The function $U_3$ is determined from (1.4) and (1.6):

$$\frac{\partial U_3}{\partial X} + \frac{1}{\alpha}U_2 + \frac{\partial V_2}{\partial Y} + \frac{\partial W_2}{\partial \zeta} = 0, \qquad U_3(0) = 0.$$

Therefore, with allowance for (2.12), we have

$$U_3 = -\frac{2}{\alpha}\left(\frac{1}{3}X^3 - \frac{1}{2}X^2\right)\frac{\partial H_1}{\partial \zeta} - \mathrm{Re}\left(\frac{1}{30}X^5 - \frac{1}{6}X^4 + \frac{2}{3}X^2\right)\frac{\partial^2 H_1}{\partial \zeta^2}$$
$$+\frac{1}{2}\overline{\mathrm{We}}\left(\frac{1}{3}X^3 - X^2\right)\left(\frac{1}{\alpha^2}\nabla^2 H_1 + \nabla^4 H_1\right) + X^2\frac{\partial H_2}{\partial \zeta} - \overline{\mathrm{Mn}}\,\mathrm{E}\,X^2\frac{\partial^3 H_1}{\partial \zeta^3} - \overline{\mathrm{Mn}}\,\mathrm{E}\,X^2\frac{\partial^3 H_1}{\partial \zeta \partial Y^2}.$$

Then, (2.13) takes the form

$$\frac{\partial H_1}{\partial T} - \frac{2}{3\alpha}\frac{\partial H_1}{\partial \zeta} - 4H_1\frac{\partial H_1}{\partial \zeta} + \frac{8}{15}\mathrm{Re}\frac{\partial^2 H_1}{\partial \zeta^2} + \frac{1}{3\alpha^2}\overline{\mathrm{We}}\nabla^2 H_1$$
$$+\overline{\mathrm{Mn}}\,\mathrm{E}\left(\frac{\partial^3 H_1}{\partial \zeta^3} + \frac{\partial^3 H_1}{\partial \zeta \partial Y^2}\right) + \frac{1}{3}\overline{\mathrm{We}}\,\nabla^4 H_1 = 0. \tag{2.14}$$

For the rotationally symmetric case,

$$\frac{\partial H_1}{\partial T} - \frac{2}{3\alpha}\frac{\partial H_1}{\partial \zeta} - 4H_1\frac{\partial H_1}{\partial \zeta} + \left(\frac{8}{15}\mathrm{Re} + \frac{1}{3\alpha^2}\overline{\mathrm{We}}\right)\frac{\partial^2 H_1}{\partial \zeta^2} + \overline{\mathrm{Mn}}\,\mathrm{E}\frac{\partial^3 H_1}{\partial \zeta^3} + \frac{1}{3}\overline{\mathrm{We}}\frac{\partial^4 H_1}{\partial \zeta^4} = 0. \tag{2.15}$$

**Remarks on the Physical Mechanism.** Equation (2.14) coincides (with accuracy up to the term with the coefficient $\overline{\mathrm{Mn}}\,\mathrm{E}$) with the equation obtained in [10]. The new term reflects the influence of thermocapillary effects on the secondary regimes in the system under study. Conditions (2.11) express the main feature of the physical mechanism that corresponds to the reduction considered. The departure of the interface from the equilibrium state changes its internal energy, thereby affecting the heat-flux perturbation through the interface. The temperature perturbation thus induced is conjugated with hydrodynamic perturbations in the process of balancing of tangential stresses on the free boundary. At the same time, perturbations of the velocity field are generated by pressure fluctuations, which, in turn, are induced by the deformation of the interface via the boundary condition of balance of normal stresses, which is intrinsic to hydrodynamic problems of this type.

A similar problem was studied in [19]. Allowance for thermocapillary effects in describing the behavior of the interface in two-phase flow of viscous immiscible liquids along a cylindrical pipe in the case where the thickness of the layer adjacent to the cylindrical wall is asymptotically small and a number of other requirements are satisfied also leads to an evolution equation of the form (2.15) where the term with the third derivative is due to thermocapillary factors.

**Wave Flow in Plane Geometry.** A thin-layer flow along a vertical plane wall is considered similarly. Let $r$ and $z$ be Cartesian coordinates (we confine ourselves to the two-dimensional case) and the film layer flow freely under gravity down the plane surface $r = a =$ const; the perturbed interface is defined by the equation $r = a + h(z,t)$. In (1.1)–(1.11), we set $v \equiv 0$ and $\partial/\partial\varphi \equiv 0$. In the initial mathematical model, Eqs. (1.1)–(1.5) are changed in an obvious manner. Now, condition (1.7) has the form

$$-\mathrm{Re}\,(p - p^g) + 2(1 + h_z^2)^{-1}(u_r - h_z(w_r + u_z) + h_z^2 w_z) = (\mathrm{We} + \mathrm{Mn}\,(\theta - 1))h_{zz}(1 + h_z^2)^{-3/2},$$

$$r = a + h(z,t).$$

The remaining boundary conditions are not changed. The basic solution has the form $u_0 = 0$, $w_0 = r^2 - 2r$, $p_0 =$ const, $\theta_0 = 1$, and $h = 1$.

An analysis similar to the one performed above, with the same conditions for the orders of magnitude of the dimensionless parameters of the problem, leads to the same chain of formulas as that in the case of cylindrical geometry, in which, formally, one should set $\alpha = \infty$. Consequently, the final amplitude equation has the form (2.15) but the terms $-(2/3\alpha)H_{1\zeta}$ and $(1/3\alpha^2)\overline{\mathrm{We}}H_{1\zeta\zeta}$ should be omitted.

**Wall-Curvature Effect on the Regularity of Secondary Motions.** As is known, the curvature of a wall down which a liquid flows can have a regularizing influence on the film-layer flow. We examine this problem in a linear approximation. Analysis of the dispersive relation of Eq. (2.14) shows [10] that the lower the wall curvature, the larger the number of linearly unstable harmonics with a high angular frequency. The thermocapillary effects considered have no effect on the reasoning in [10] because of their purely dispersive character.

**3. Formation of Quasistationary Regularized Regimes.** We confine ourselves to the rotationally symmetric case. By substituting variables, we reduce Eq. (2.15) to the form

$$H_t + 2QHH_x + UH_{xx} + DH_{xxx} + SH_{xxxx} = 0 \qquad (U > 0, \quad S > 0). \tag{3.1}$$

The linear dispersive relation of Eq. (3.1) for the harmonic $\propto \exp(\lambda t + inx)$ has the form $\lambda = Un^2 + iDn^3 - Sn^4$. Hence, the term with the second derivative is due to destabilizing effects, the term with the third derivative to dispersive effects, and the term with the fourth derivative to dissipative effects. Relation (3.1) is one of the simplest models of weakly nonlinear dissipative processes in systems with dispersive and destabilizing factors. This equation is used for various problems, for example, in describing the hydrodynamics of film flows [20], in plasma physics, and in studies of modulations of the processes described by the generalized Landau–Ginzburg equation [21].

Let us give some known results of the numerical studies in [9, 11–16] of the general properties of $L$-periodic solutions described by Eq. (3.1) $[H(t,0) = H(t,L)]$. Below, it will be convenient to use the parameter $\mu = (L/2\pi)(U/S)^{1/2} > 0$, which reflects the level of instability of the equilibrium state and appears naturally when the problem considered is reduced to its canonical form.

For $D = 0$, (3.1) is the Kuramoto–Sivashinskii equation, which is often mentioned as the simplest deterministic model of weak turbulence at interfaces and fronts [12]. For comparatively small values of the parameter $\mu$, regular solutions of either steady-state or oscillating character are observed, but with increase in the spatial period on the axis of the bifurcation parameter $\mu$, spots of irregular behavior appear, and for sufficiently large values of $\mu$, distinct chaotic behavior is established (as attractors or quite long transition processes) [9, 12, 14]. These irregular regimes are chaotic oscillations in time with components of spatial coherence.

When the value of $\mu$ is large so that the dynamics of the Kuramoto–Sivashinskii equation is of a chaotic character, and the positive values of the coefficient $D$ are fairly close to zero, stochastic behavior is observed. However, with an increase in $D$, the level of regularization increases sharply: one can observe sequences of solitary waves, and with the development of dispersive effects, the distances between individual pulses change in a more and more regular manner. Finally, for rather large values of the parameter $D$, a train of solitary waves with practically the same profiles and distances between neighboring maxima is formed after a transient process [14–16].

Thus, under the influence of dispersive effects, the irregular regimes of the Kuramoto–Sivashinskii equation become quasistationary regular.

Under certain conditions, the limiting regimes of Eq. (3.1) established after transient unsteady processes (sequences of solitary waves) can be satisfactorily described by an asymptotic solution that is a superposition of stationary solitary waves with one maximum $[H^s(x - Vt + \psi_i)$, where $H^s \to 0$ as $x \to \pm\infty]$ which interact with each other rather weakly [16, 22]. The profile of $H^s$ is determined numerically. In such a train of identical solitary waves, the distances between pulses can vary in both regular and irregular manners [16, 22].

Equation (3.1) also has other types of steady solutions $H(x - Vt)$, which correspond qualitatively to the most important types of regular film flows: periodic regimes and sequences of solitary waves with several maxima. In the phase space $(H, H', H'')$, the latter correspond to nearly biasymptotic trajectories to a stationary point $(0, 0, 0)$ that return cyclically to it and, in each cycle, make a few rotations about the second stationary point $(V/Q, 0, 0)$ [22, 23].

Equation (3.1) can also describe regimes of the type of hydraulic jumps. In a special case, the corresponding exact solution (by analogy with the Kuramoto–Sivashinskii equation) [24] can be obtained. That is, the equation

$$-VH + QH^2 + UH' + DH'' + SH''' = q = \text{const} \tag{3.2}$$

$[H = H(\xi)$, where $\xi = x - Vt]$ has a solution of the form

$$H = a + b\tanh(\alpha\xi) + c\tanh^2(\alpha\xi) + d\tanh^3(\alpha\xi). \tag{3.3}$$

Here

$$a = \frac{V}{2Q} + \frac{15}{2}\delta M, \quad b = \left(\frac{180}{7} + \frac{15}{56}\delta^2\right)M, \quad c = -\frac{15}{2}\delta M, \quad d = 60M, \quad M = S\alpha^3/Q \tag{3.4}$$

if the conditions

$$V^2 = 4\left(\frac{83\,520}{49} + \frac{3000}{49}\delta^2 + \frac{195}{3136}\delta^4\right)S^2\alpha^6 - 4Qq, \quad U = \left(\frac{380}{7} + \frac{13}{56}\delta^2\right)S\alpha^2, \quad D = \delta S\alpha \tag{3.5}$$

are satisfied ($\delta = \pm 32$ or $\delta = \pm 24$).

In the phase space, solution (3.3)–(3.5) of Eq. (3.2) corresponds to a heteroclinic orbit.

Finally, let us briefly consider how the dynamics is affected by the dispersive effects in the case where the value of the parameter $\mu$ is sufficiently small so that for $D = 0$, Eq. (3.1) has regular periodic solutions. The problem was studied numerically by the Galerkin technique with a number of approximating modes of the order of $5\mu$ [25]. Numerical calculations showed the following.

1. Attractors of the Kuramoto–Sivashinskii equation ($D = 0$) of a unimodal steady-state form [11] are transformed into steady traveling waves with a single extremum on the spatial period.

2. Within the range of values of $\mu$ where the limiting regime for $D = 0$ is periodic pulsations between two steady states that are invariant with respect to $\pi$-translations [13], pulsating traveling waves can occur for sufficiently weak dispersion. The form of these wave formations is periodically changed: the first and second Fourier components dominate alternately. If the dispersion is relatively high, the limiting regime is a sequence

of traveling waves of constant form.

3. Within the range of values of $\mu$ where bimodal steady states [11] are attractors of the Kuramoto–Sivashinskii equation, the following types of behavior are established by numerical calculations:

(a) traveling waves of constant form characterized by the presence of two extrema in the spatial period;

(b) attractors of the form of traveling waves pulsating with time periodically;

(c) trains of traveling waves with a single extremum on the period.

Thus, the occurrence of dispersive effects extends the variety of limiting regimes for small values of the parameter $\mu$ corresponding to the regular attractors of the Kuramoto–Sivashinskii equation.

**Conclusions.** The essence of the physical mechanism proposed here for the development of longwave perturbations in a thin liquid layer is that variations in the internal energy of the liquid–gas interface cause corresponding variations in the temperature field in the vicinity of the interface, thereby inducing (or changing) the Marangoni stresses. The same feature is intrinsic to the thermocapillary mechanism considered in [19] for a liquid–liquid system.

In the examined case of linear dependence of the surface-tension coefficient on temperature, the specific (referred to the interface unit area) surface internal energy is constant [17]. However, the internal energy of the interface can vary owing to surface area fluctuations.

Let us consider whether the thermocapillary mechanism of perturbation propagation is realizable. The key parameter for the analysis is $\mathrm{Mn}\,\mathrm{E}$. This parameter expresses the comparative magnitude of the effects caused by the presence of surface internal energy and the Marangoni effect. It is known [1] that, usually, the value of $\mathrm{Mn}\,\mathrm{E}$ is small but at high temperatures, $\mathrm{Mn}\,\mathrm{E} = O(1)$ for most liquids. The value of $\mathrm{Mn}\,\mathrm{E}$ also increases for low-viscosity liquids. In the case considered here, $\mathrm{Mn}\,\mathrm{E} = O(\varepsilon^{-1})$ and in [19], $\mathrm{Mn}\,\mathrm{E} = O(1)$.

Use of (1.10), as the condition of energy transfer through the free boundary, instead of the frequently used condition of the form

$$k\nabla\theta \cdot \mathbf{n} + b(\theta - \theta^g) = 0, \qquad \mathbf{x} \in \Gamma,$$

where $b(\mathbf{x}, t)$ is the coefficient of interphase heat exchange and $\theta^g$ is the controlled temperature at a point of the gas phase, can lead to known difficulties in performing experiments according to the mathematical formulation of the problem. However, in the case considered, where the basic state of the system is isothermal, this problem will not apparently be an obstacle (see corresponding explanations in Sec. 2).

The foregoing suggests that the physical mechanism considered here can be realized in practice for liquid–gas or liquid–liquid systems.

In this connection, it is worth noting that the wave regimes of film flow observed in experimental studies can qualitatively correspond to solutions of simplified models similar to Eq. (2.15) over wider ranges of the determining parameters than is specified in the derivation of these models.

In conclusion, we note two characteristic features of the influence of the examined thermocapillary effects on the wave regimes in a thin layer. In the approximation considered, the Marangoni stresses related to the variation in the internal energy of the layer free surface facilitate:

(a) regularization of dynamics;

(b) formation of wave regimes of the type of either irregular or quasiregular sequence of solitary waves.

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